Assignment 6

Hand in no. 4 and 5 by October 31, 2024.

1. Let $f : E \to Y$ be a uniformly continuous map where $E \subset X$ and X, Y are metric spaces. Suppose that Y is complete. Show that there exists a uniformly continuous map F from $\overline{E}$ to Y satisfying $F = f$ in E. In other words, f can be extended to the closure of E preserving uniform continuity.

2. Let $A = \{a_{ij}\}$ be an $n \times n$ matrix. Show that
$$\|Ax\| \leq \sqrt{\sum_{i,j} a_{ij}^2}\ \|x\|.$$

3. Can you solve the system of equations
$$x + y^4 = 0, \quad y - x^2 = 0.015\ ?$$

4. Can you solve the system of equations
$$x + y - x^2 = 0, \quad x - y + xy\sin y = -0.002\ ?$$
Hint: Put the system in the form $x + \cdots = 0, \quad y + \cdots = 0$, first.

5. Let $A = (a_{ij})$ be an $n \times n$ matrix. Show that the matrix $I + A$ is invertible if $\sum_{i,j} a_{ij}^2 < 1$. Give an example showing that $I + A$ could become singular when $\sum_{i,j} a_{ij}^2 = 1$.

6. Consider the iteration
$$x_{n+1} = \alpha x_n(1 - x_n),\ x_0 \in [0, 1]\ .$$
Find
   (a) The range of α so that $\{x_n\}$ remains in $[0, 1]$.
   (b) The range of α so that the iteration has a unique fixed point 0 in $[0, 1]$.
   (c) Show that for $\alpha \in [0, 1]$ the fixed point 0 is attracting in the sense: $x_n \to 0$ whenever $x_0 \in [0, 1]$.